Nucleon-Deuteron Scattering from an Effective Field Theory

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Abstract

We use an effective field theory to compute low-energy nucleon-deuteron scattering. We obtain the quartet scattering length using low energy constants entirely determined from low-energy nucleon-nucleon scattering. We find $a_{th}=6.33$ fm, to be compared to $a_{exp}=6.35\pm0.02$ fm.

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There has been considerable interest lately in a description of nuclear forces from the low-energy effective field theory (EFT) of QCD. (For a review, see Ref. [1].) Following a program suggested by Weinberg [2], the leading components of the nuclear potential have been derived [3] and a reasonable fit to two-nucleon properties has been achieved [4]. The correct formulation of the nuclear force problem within the EFT method is important because it will allow a systematic calculation of nuclear properties consistently with QCD. One would like, for example, to devise a theory of nuclear matter rooted in a hadronic theory that treats chiral symmetry correctly and yields the well-known few-nucleon phenomenology. One hopes that after a number of parameters of the EFT are either calculated from first principles or fitted to a set of few-nucleon data, the theory can be used to predict other reactions involving light nuclei and features of heavier nuclei.

However, some issues concerning renormalization in this non-perturbative context and fine-tuning in the two-nucleon S-waves have been raised in Refs. [5, 6] and are still not fully understood [7]. The fine-tuning necessary to bring a (real or virtual) bound state very close to threshold generates a scattering length a much larger than other scales in the problem. At momenta of O(1/a), mesons can be integrated out and the characteristic mass scale μ of the underlying theory controls the size of the other effective range parameters; for example, the effective range $r_0 \sim 2/\mu$. Once the leading order contributions, which give rise to a, are included to all orders, the EFT at momenta O(1/a) becomes an expansion in powers of $1/(a\mu)$. Kaplan [6] has noticed that the interactions that generate a non-zero r_0 can also be resummed by the introduction of a baryon number two state of mass $\Delta = 2/Ma|r_0|$, which in lowest order in a derivative expansion couples to two nucleons with a strength $g^2/4\pi = 1/M^2|r_0|$. In Ref. [6] it was shown how this works in the two-nucleon ${}^{1}S_{0}$ channel. Analogous considerations hold for the ${}^{3}S_{1}$ channel, where they are similar to the old quasi-particle approach of Weinberg [8].

In this paper we consider the application of these ideas to the three-nucleon system. Our goal here is to calculate some of the three-nucleon parameters that are dominated by the leading interactions in the EFT without pions. We show in particular that the quartet scattering length in neutron-deuteron scattering can be predicted once the EFT is constrained by low-energy two-nucleon data. Such an attempt to a model-independent or "universal" approach is not a new idea; it permeates for example the work of Efimov (see, e.g., [9]) and Amado (see, e.g., [10]). However, as we will show,

the EFT formulation is much easier to implement, from both conceptual and practical standpoints.

For momenta of order 1/a (the momentum scale relevant for zero-energy Nd scattering), we can integrate out mesons and consider an EFT with only nucleons N. Interactions are then described by a tower of nucleon contact operators with an increasing number of derivatives. Amplitudes in leading order are given by a zero-range four-nucleon interaction iterated to all orders. Corrections come in powers of $1/(am_{\pi})$. The next two orders in this expansion, $1/(am_{\pi}) \sim r_0/2a$ and $1/(am_{\pi})^2 \sim (r_0/2a)^2$, stem from one and two insertions of a two-derivative four-nucleon operator giving rise to a non-zero r_0 . It is advantageous to sum all the contributions coming from this operator, which can be easily done because they appear in a geometric series. The resulting interaction is equivalent to the s-channel propagation of a dibaryon, and therefore can be obtained more directly by the introduction of a dibaryon field. Assuming naturalness, only higher orders depend on further NN scattering parameters —such as the shape parameter, which contributes at $O(1/(am_{\pi})^3)$ — and three-nucleon forces, which start at $O(1/(am_{\pi})^4)$.

Since in both I=0 and I=1 S-wave two-nucleon channels we observe (one real, one virtual) bound states near threshold, we consider two dibaryon fields, $T(\vec{D})$ of spin zero (one) and isospin one (zero). The most general Lagrangian invariant under parity, time-reversal, and small Lorentz boosts is

$$\mathcal{L} = N^{\dagger} (i\partial_{0} + \frac{\vec{\nabla}^{2}}{2M} + \ldots)N$$

$$+ \mathbf{T}^{\dagger} \cdot (-i\partial_{0} - \frac{\vec{\nabla}^{2}}{4M} + \Delta_{T} + \ldots)\mathbf{T} + \vec{D}^{\dagger} \cdot (-i\partial_{0} - \frac{\vec{\nabla}^{2}}{4M} + \Delta_{D} + \ldots)\vec{D}$$

$$- \frac{g_{T}}{2} (\mathbf{T}^{\dagger} \cdot N\sigma_{2}\boldsymbol{\tau}\tau_{2}N + \text{h.c.}) - \frac{g_{D}}{2} (\vec{D}^{\dagger} \cdot N\tau_{2}\vec{\sigma}\sigma_{2}N + \text{h.c.}) + \ldots$$
(1)

Here the $\Delta_{T,D}$ and $g_{T,D}$ are undetermined parameters and "..." stands for higher order terms. (Note that the effects of non-derivative and two-derivative four-nucleon terms can be absorbed into a redefinition of $\Delta_{T,D}$ and $g_{T,D}$ and higher order four-nucleon terms.)

In this non-relativistic theory all particles propagate forward in time, nucleon tadpoles vanish and, as a consequence, there is no dressing of the



Figure 1: Dressed dibaryon propagator.

nucleon propagator, which is simply

$$S_N(p) = \frac{i}{p^0 - \frac{\vec{p}^2}{2M} + i\epsilon}.$$
 (2)

The propagators for dibaryons are more complicated, because of the coupling to two-nucleon states. The dressed propagators consist of the bubble sum in Fig. 1, which amounts to a self-energy contribution proportional to the bubble integral. This integral is proportional to the (large) mass M, and it is this enhancement that gives rise to non-perturbative phenomena and leads eventually to the existence of bound states. The integral is also ultraviolet divergent and requires regularization. Introducing a cut-off Λ we find a linear divergence $\propto \Lambda$, a cut-off independent piece which is non-analytic in the energy, a term that goes as Λ^{-1} and terms that are higher order in Λ^{-1} . The first and third terms can be absorbed in renormalization of the parameters of the Lagrangian (1); in what follows we omit a label R that should be attached to these parameters, i.e., $\Delta_{T,D}$ and $g_{T,D}$ stand for the renormalized parameters. Higher order terms are neglected because they are of the same order as interactions in the "..." of the Lagrangian (1). A dibaryon propagator has therefore the form

$$iS_D(p) = \frac{1}{p^0 - \frac{\vec{p}^2}{4M} - \Delta_D + \frac{M^2 g_D^2}{2\pi} \sqrt{-Mp^0 + \frac{\vec{p}^2}{4} - i\epsilon} + i\epsilon}.$$
 (3)

Note that such a dressed propagator has two poles at $p^0 = \vec{p}^2/4M - B$, $p^0 = \vec{p}^2/4M - B_{deep}$ and a cut along the positive real axis starting at $p^0 = \vec{p}^2/4M$.

The NN amplitude can now be obtained directly from $S_D(p)$ as in Fig. 2. In the center-of-mass, the on-shell I=0, J=1 S-wave amplitude at an energy $E=k^2/M$ is

$${}^{3}T_{NN}(k) = \frac{4\pi}{M} \frac{1}{-\frac{2\pi\Delta_{D}}{Mq_{D}^{2}} + \frac{2\pi}{M^{2}q_{D}^{2}}k^{2} - ik},$$
(4)



Figure 2: NN amplitude.

which is exactly equivalent to the effective range expansion. An analogous result holds for the I=1 S-wave. The four parameters $\Delta_{T,D}$ and $g_{T,D}$ can then be fixed from the experimentally known scattering lengths and effective ranges. The NN amplitude has shallow poles at $B \sim 1/Ma^2$ which are associated with the deuteron in the 3S_1 channel and with the virtual bound state in the 1S_0 channel. The effective theory has also an additional deep bound state in each channel at $B_{deep} \sim 4/Mr_0^2$, which is outside the range of validity of the EFT.

From the triplet parameters $^3a=5.42$ fm and $^3r_0=1.75$ fm [11] we find $\Delta_D=8.7$ MeV and $g_D^2=1.6\cdot 10^{-3}$ MeV. The resulting deuteron binding energy is B=2.28 MeV. From the singlet parameters $^1a_{pp}=-17.3$ fm, $^1a_{np}=-23.75$ fm, $^1a_{nn}=-18.8$ fm, $^1r_{0pp}=2.85$ fm, $^1r_{0np}=2.75$ fm, and $^1r_{0nn}=2.75$ fm [12] we find the averages $\Delta_T=-1.5$ MeV and $g_T^2=1.0\cdot 10^{-3}$ MeV.

With the parameters so determined, we turn now to possible predictions in low-energy nucleon-deuteron scattering. For simplicity we restrict ourselves to scattering below the deuteron break-up threshold, where the Swave is dominant. There are two S-wave channels, corresponding to total spin J = 3/2 and J = 1/2. In the quartet only \vec{D} contributes while in the doublet T also appears. The Nd scattering amplitude T_{Nd} from the same interactions is given by the diagrams in Fig. 3, which can be summed up by solving an integral equation in the quartet and a pair of coupled integral equations in the doublet channel. The diagrams under consideration are power-counting finite, but this does not preclude the existence of relevant contact interactions between the nucleon and the dibaryons. Pion exchange that would generate such interactions can be expected to be larger for the I=1 dibaryon T, and therefore predominantly affect the J=1/2 channel. We will return to the doublet case in a future publication. Here we study the quartet channel expected to be much less sensitive to the details of the physics of momenta of $O(m_{\pi})$, since the wave function on this channel

Figure 3: Nd amplitude.

vanishes by symmetry when the three particles are at the same point.

An enormous simplification comes about because the s-channel interaction due to the dibaryon is both local and separable. This allows us to write a simple integral equation that sums all the graphs in Fig. 3. Performing the integration over the time-component of the loop 4-momentum, we find that the conveniently normalized on-shell amplitude as a function of the initial (final) center-of-mass 3-momentum \vec{k} (\vec{p}) satisfies

$$\left[\frac{3(\vec{p}^2 - \vec{k}^2)}{8M^2 g_D^2} + \frac{1}{4\pi} \left(\sqrt{\frac{3}{4}} (\vec{p}^2 - \vec{k}^2) + MB - \sqrt{MB} \right) \right] \frac{t(\vec{p}, \vec{k})}{\vec{p}^2 - \vec{k}^2 - i\epsilon}$$

$$= \frac{-1}{(\vec{p} + \vec{k}/2)^2 + MB} - \int \frac{d^3l}{(2\pi)^3} \frac{1}{\vec{l}^2 + \vec{l} \cdot \vec{p} + \vec{p}^2 - \frac{3}{4}\vec{k}^2 + MB} \frac{t(\vec{l}, \vec{k})}{\vec{l}^2 - \vec{k}^2 - i\epsilon}.$$
(5)

Note that all terms in a perturbative expansion of t in (5) are of the same order ($\sim 1/\sqrt{MB}$). It is straightforward but tedious to show that the wave function $(2\pi)^3 \delta(\vec{p}-\vec{k}) + t(\vec{p},\vec{k})/(\vec{p}^2-\vec{k}^2-i\epsilon)$ corresponding to a scattering solution indeed satisfies the Schrödinger equation derived from the Lagrangian (1).

At zero energy $(k \to 0)$ only the S-wave, depending on the magnitudes of momenta, contributes to the scattering, and we can perform the angular integration directly. It is also convenient to normalize all quantities to \sqrt{MB} . Defining $\vec{x} = \vec{p}/\sqrt{MB}$,

$$a(x) = \frac{\sqrt{MB}}{4\pi} t(\frac{p}{\sqrt{MB}}, 0), \tag{6}$$

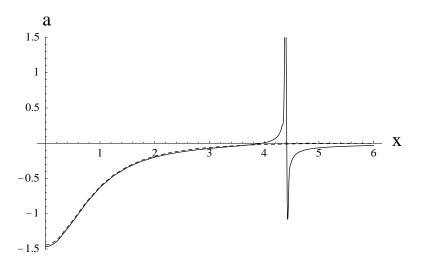


Figure 4: Function a(x) for $\eta = 0.40$ without (solid line) and with (dashed line) cut-off.

and introducing

$$F(x,z) = \frac{1}{xz} \ln(\frac{x^2 + z^2 + 1 + xz}{x^2 + z^2 + 1 - xz}),\tag{7}$$

Eq. (5) becomes

$$\frac{3}{4} \left[-\eta + \frac{1}{1 + \sqrt{1 + \frac{3}{4}x^2}} \right] a(x) = -\frac{1}{x^2 + 1} - \frac{1}{\pi} \int_0^\infty dz F(x, z) a(z). \tag{8}$$

Note that there is only one parameter $\eta = 2\pi\sqrt{MB}/M^2g_D^2 = {}^3r_0\sqrt{MB}/2 = 0.40$ in this equation. The value of the function a(x) at x=0 gives the Nd scattering length in units of $1/\sqrt{MB}$. The same equation was previously obtained and solved in the zero-range limit $(\eta \to 0)$ [13].

We have solved Eq. (8) numerically for $\eta = 0.40$ by the Nystrom method [14]. The solution a(x) is plotted as the solid line in Fig. 4. The pole in a(x) around $x \sim 4.4$ is associated with the spurious deep two-body pole. Its presence allows intermediate states where two nucleons fall into this deep state while the other has extra energy. This means that the outgoing wave has an additional component, a pole at the momentum corresponding to this additional process. The interesting point is that even though the effective

theory makes nonsensical predictions outside its domain of validity, like the existence of this new state in Nd scattering, the low-x part of the curve is insensitive to the large-x behavior, and the prediction for the scattering length is sensible. In order to demonstrate this more explicitly we have also solved Eq. (8) with a cut-off two-nucleon amplitude without the deep pole. For a cut-off of 150 MeV we obtain the broken line in Fig. 4.

The quartet scattering length is ${}^4a = -a(0)/\sqrt{MB}$. For $\eta = 0$ (and B fixed), we reproduce the result ${}^4a = 5.09$ fm of Ref. [13]. Taking into account the finite range ($\eta = 0.40$) we obtain (Fig. 4) ${}^4a = 6.33$ fm with an uncertainty from higher orders of $\sim \pm 0.10$ fm. This result obtained with no free parameters is in very good agreement with the experimental value of ${}^4a = 6.35 \pm 0.02$ fm [15].

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